Differentiation exercise – show differential equation

1. If $y = x\sin 2x$, prove that $x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + \frac{2y}{x} + 4xy = 0$

$y = x\sin 2x \Longrightarrow \frac{dy}{dx} = \sin 2x + 2x\cos 2x$

$\frac{d^2y}{dx^2} = 2\cos 2x + (2\cos 2x - 4x\sin 2x)$

$x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + \frac{2y}{x} + 4xy$

$= (2x\cos 2x + 2x\cos 2x - 4x^2\sin 2x) - (2\sin 2x + 4x\cos 2x) + \dfrac{2x\sin 2x}{x} + 4x^2\sin 2x = 0$

2. Given that $y = e^x - e^{-x}$, show that $\left(\frac{dy}{dx}\right)^2 - y^2 - 4 = 0$

$y = e^x - e^{-x} \Longrightarrow \frac{dy}{dx} = e^x + e^{-x}$

$\left(\frac{dy}{dx}\right)^2 - y^2 - 4 = (e^x + e^{-x})^2 - (e^x - e^{-x})^2 - 4 = (e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x}) - 4 = 0$

3. Given that $v = \sqrt{\sin u}$, show that $4v^3\frac{d^2v}{du^2} + v^4 + 1 = 0$

$\frac{dv}{du} = \frac{\cos u}{2\sqrt{\sin u}}$

$\frac{d^2v}{du^2} = \frac{1}{2}\frac{\sqrt{\sin u}(-\sin u) - \cos u\frac{\cos u}{2\sqrt{\sin u}}}{\sin u} = \frac{1}{4}\frac{-2\sin^2 u - \cos^2 u}{\sin u\sqrt{\sin u}} = -\frac{\sin^2 u + 1}{4\sin u\sqrt{\sin u}} = -\frac{v^4+1}{4v^3}$

$\therefore 4v^3\frac{d^2v}{du^2} + v^4 + 1 = 0$

4. Given $y = e^{-x}\cos x$, show that $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0.$

Method 1

$y = e^{-x}\cos x$

$\frac{dy}{dx} = -e^{-x}\sin x - e^{-x}\cos x$

$\frac{d^2y}{dx^2} = (e^{-x}\sin x - e^{-x}\cos x) + (e^{-x}\sin x + e^{-x}\cos x) = 2e^{-x}\sin x$

$\therefore \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = (2e^{-x}\sin x) - 2(e^{-x}\sin x + e^{-x}\cos x) + 2e^{-x}\cos x = 0$

Method 2

$y = e^{-x}\cos x \implies \ln y = -x + \ln(\cos x)$

$\frac{1}{y}\frac{dy}{dx} = -1 - \frac{\sin x}{\cos x}$

$\frac{dy}{dx} = -y - y\tan x$

$\frac{dy}{dx} + 2y = y(1 - \tan x)$

Differentiate, we get: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = \frac{dy}{dx}(1 - \tan x) - y\sec^2 x$

$= (-y - y\tan x)(1 - \tan x) - y\,(1 + \tan^2 x)$

$= -2y$

$\therefore \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

Method 3

$y = e^{-x}\cos x \implies y\,e^{x} = \cos x \quad \ldots (1)$

$\frac{dy}{dx}e^{x} + ye^{x} = -\sin x$

$\left(\frac{d^2y}{dx^2}e^{x} + \frac{dy}{dx}e^{x}\right) + \left(\frac{dy}{dx}e^{x} + ye^{x}\right) = -\cos x$

$\frac{d^2y}{dx^2}e^{x} + 2\frac{dy}{dx}e^{x} + ye^{x} = -ye^{x}$, by (1)

$\therefore \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

5. Given that $y = \frac{\sin kx}{1+\cos kx}$, where k is a positive integer, show that $\sin kx\,\frac{d^2y}{dx^2} = k^2y^2$.

Method 1

$\frac{dy}{dx} = \frac{(1+\cos kx)(k\cos kx) - \sin kx(-k\sin kx)}{(1+\cos kx)^2} = \frac{k\cos kx + k\cos^2 kx + k\sin^2 kx}{(1+\cos kx)^2} = \frac{k\cos kx + k}{(1+\cos kx)^2} = \frac{k(1+\cos kx)}{1+\cos kx} = \frac{k}{1+\cos kx}$

$\sin kx\,\frac{dy}{dx} = k\frac{\sin kx}{1+\cos kx} = ky$

$\sin kx\,\frac{d^2y}{dx^2} + k\cos kx\,\frac{dy}{dx} = k\frac{dy}{dx}$

$\sin kx\,\frac{d^2y}{dx^2} = k\frac{dy}{dx}(1 - \cos kx) = k\frac{dy}{dx}\frac{(1-\cos kx)(1+\cos kx)}{(1+\cos kx)} = k\frac{dy}{dx}\frac{\sin^2 kx}{(1+\cos kx)} = k\left[\sin kx\,\frac{dy}{dx}\right]\frac{\sin kx}{1+\cos kx}$

$= k[ky]y = k^2y^2$

Method 2

Given that $y = \frac{\sin kx}{1+\cos kx}$, where k is a positive integer, show that $\sin kx \frac{d^2y}{dx^2} = k^2y^2$.

Note that : $y = \frac{\sin kx}{1+\cos kx} = \frac{1-\cos kx}{\sin kx}$

$$\frac{dy}{dx} = \frac{\sin kx(k\sin kx)-(1-\cos kx)(k\cos kx)}{\sin^2 kx} = \frac{k\sin^2 kx+k\cos^2 kx-k\cos kx}{\sin^2 kx} = \frac{k-k\cos kx}{\sin^2 kx} = \frac{k}{\sin kx}\frac{1-\cos kx}{\sin kx} = \frac{ky}{\sin kx}$$

$$\sin kx\frac{dy}{dx} = ky$$

$$\sin kx\frac{d^2y}{dx^2} + k\cos kx\frac{dy}{dx} = k\frac{dy}{dx}$$

$$\sin kx\frac{d^2y}{dx^2} = k(1-\cos kx)\frac{dy}{dx} = k\left(\frac{1-\cos kx}{\sin kx}\right)\left(\sin kx\frac{dy}{dx}\right) = k(y)(ky) = k^2y^2$$

6. Given $y(2 - x) = 3$, show that $3\frac{d^2y}{dx^2} - 2y\frac{dy}{dx} = 0$.

Method 1

Differentiate, $(2 - x)\frac{dy}{dx} - y = 0$ $\frac{dy}{dx} = \frac{y}{2-x}$

Differentiate again, $\frac{d^2y}{dx^2} = \frac{(2-x)\frac{dy}{dx}-y(-1)}{(2-x)^2} = \frac{(2-x)\frac{y}{2-x}+y}{(2-x)^2} = \frac{2y}{(2-x)^2}$

$$3\frac{d^2y}{dx^2} - 2y\frac{dy}{dx} = 3\frac{2y}{(2-x)^2} - 2y\frac{y}{2-x} = \frac{6y}{(2-x)^2} - \frac{2y^2}{2-x} = \frac{6y}{(2-x)^2} - \frac{2y[y(2-x)]}{(2-x)^2} = \frac{6y}{(2-x)^2} - \frac{2y[3]}{(2-x)^2} = 0$$

Method 2

Differentiate, $(2 - x)\frac{dy}{dx} - y = 0$

Differentiate again, $\left[(2 - x)\frac{d^2y}{dx^2} - \frac{dy}{dx}\right] - \frac{dy}{dx} = 0$ or $(2 - x)\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$

Multiply by y, $y(2 - x)\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$

$$\therefore 3\frac{d^2y}{dx^2} - 2y\frac{dy}{dx} = 0$$

7. Given $y = (1 + 4x)e^{-2x}$, prove that $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

Method 1

$$\frac{dy}{dx} = 2e^{-2x} - 8xe^{-2x}$$

$$\frac{d^2y}{dx^2} = 16xe^{-2x} - 12e^{-2x}$$

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = (16xe^{-2x} - 12e^{-2x}) + 4(2e^{-2x} - 8xe^{-2x}) + 4(1+4x)$$

$$= (16xe^{-2x} - 12e^{-2x}) + (8e^{-2x} - 32xe^{-2x}) + 4e^{-2x} + 16xe^{-2x} = 0$$

Method 2

$$e^{2x}y = (1+4x)$$

$$e^{2x}\frac{dy}{dx} + 2ye^{2x} = 4$$

$$\left(e^{2x}\frac{d^2y}{dx^2} + 2e^{2x}\frac{dy}{dx}\right) + \left(2e^{2x}\frac{dy}{dx} + 4e^{2x}y\right) = 0$$

$$e^{2x}\left(\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y\right) = 0$$

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

8. Let $y = \sqrt{\cos x}$, show that $4y^3\frac{d^2y}{dx^2} + y^4 + 1 = 0$.

$y = \sqrt{\cos x} \Rightarrow y^2 = \cos x$

Differentiate, $2y\frac{dy}{dx} = -\sin x \quad \dots (1)$

Differentiate again, $2y\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{dy}{dx} = -\cos x = -y^2$

$$2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = -y^2$$

Multiply by $2y^2$, $4y^3\frac{d^2y}{dx^2} + \left(2y\frac{dy}{dx}\right)^2 = -2y^4$

By (1), $4y^3\frac{d^2y}{dx^2} + (-\sin x)^2 = -2y^4$, $4y^3\frac{d^2y}{dx^2} + 1 - \cos^2 x = -2y^4$

$$4y^3\frac{d^2y}{dx^2} + 1 - y^4 = -2y^4$$

$$4y^3\frac{d^2y}{dx^2} + y^4 + 1 = 0$$

9. Given $(1+x^2)y^2 = 1 - x^2$, show that $\left(\frac{dy}{dx}\right)^2 = \frac{1-y^4}{1-x^4}$.

Method 1

$(1+x^2)y^2 = 1 - x^2 \quad \dots (1)$

$$y^2 = \frac{1-x^2}{1+x^2},\quad 1 - y^2 = 1 - \frac{1-x^2}{1+x^2} = \frac{2x^2}{1+x^2},\quad 1 + y^2 = 1 + \frac{1-x^2}{1+x^2} = \frac{2}{1+x^2}$$

$$1 - y^4 = (1-y^2)(1+y^2) = \frac{4x^2}{(1+x^2)^2}$$

Differentiate (1), $2xy^2 + (1+x^2)\left[2y\frac{dy}{dx}\right] = -2x$

$$\frac{dy}{dx}=\frac{-2x(1+y^2)}{2y(1+x^2)},\quad \left(\frac{dy}{dx}\right)^2=\frac{4x^2}{(1+x^2)^2}\frac{(1+y^2)^2}{4y^2}=(1-y^4)\frac{\left(\frac{2}{1+x^2}\right)^2}{4\left(\frac{1-x^2}{1+x^2}\right)}=\frac{1-y^4}{(1+x^2)(1-x^2)}=\frac{1-y^4}{1-x^4}$$

Method 2

Let $x=\tan\frac{\theta}{2},\ \cos\theta=\frac{1-x^2}{1+x^2},\ \frac{dx}{d\theta}=\frac{1}{2}(1+x^2)$

$$y^2=\frac{1-x^2}{1+x^2}=\cos\theta\,,\quad y=\sqrt{\cos\theta}$$

$$\frac{dy}{d\theta}=-\frac{\sin\theta}{2\sqrt{\cos\theta}},\quad \left(\frac{dy}{d\theta}\right)^2=\frac{\sin^2\theta}{4\cos\theta}=\frac{1-\cos^2\theta}{4\cos\theta}=\frac{1-y^4}{4\left(\frac{1-x^2}{1+x^2}\right)}$$

$$\left(\frac{dy}{dx}\right)^2=\frac{\left(\frac{dy}{d\theta}\right)^2}{\left(\frac{dx}{d\theta}\right)^2}=\frac{\frac{1-y^4}{4\left(\frac{1-x^2}{1+x^2}\right)}}{\left[\frac{1}{2}(1+x^2)\right]^2}=\frac{1-y^4}{(1+x^2)(1-x^2)}=\frac{1-y^4}{1-x^4}$$

10. Form a differential equation from $y=Ax^3+\frac{B}{x^2}-6x,\ x>0$.

$yx^2=Ax^5+B-6x^3$

Differentiate, $x^2\frac{dy}{dx}+2xy=5Ax^4-18x^2$

Divide by x, $x\frac{dy}{dx}+2y=5Ax^3-18x \quad \ldots(1)$

Differentiate (1), $\left(x\frac{d^2y}{dx^2}+\frac{dy}{dx}\right)+\left(2\frac{dy}{dx}\right)=15Ax^2-18$

$$x\frac{d^2y}{dx^2}+3\frac{dy}{dx}=15Ax^2-18$$

Multipy by x, $x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}=15Ax^3-18x$

$$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}=3(5Ax^3-18x)+36x=3\left(x\frac{dy}{dx}+2y\right)+36x\ \text{, by (1)}$$

$$x^2\frac{d^2y}{dx^2}-6xy=36x$$

$$x\frac{d^2y}{dx^2}-6y=36$$

11. Form a differential equation from $y=Ax^3+\frac{B}{x^2}-6,\ x>0$.

$yx^2=Ax^5+B-6x^2$

Differentiate, $x^2\frac{dy}{dx}+2xy=5Ax^4-12x$

Divide by x, $x\frac{dy}{dx} + 2y = 5Ax^3 - 12 \quad \ldots (1)$

Differentiate (1), $\left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) + \left(2\frac{dy}{dx}\right) = 15Ax^2$

$$x\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 15Ax^2$$

Multipy by x, $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} = 15Ax^3$

$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} = 3(5Ax^3 - 12) + 36 = 3\left(x\frac{dy}{dx} + 2y\right) + 36$, by (1)

$$x^2\frac{d^2y}{dx^2} - 6y = 36$$

$$x^3\frac{d^2y}{dx^2} - 6xy = 36x$$

12. $y\sin^{-1}3x = \sqrt{1 - 9x^2}$, show that $(1 - 9x^2)\frac{dy}{dx} + 3y^2 + 9xy = 0$

$$y\sin^{-1}3x = \sqrt{1 - 9x^2}$$

$$\sin^{-1}3x\frac{dy}{dx} + y\frac{3}{\sqrt{1-9x^2}} = \frac{-18x}{2\sqrt{1-9x^2}}$$

$$y\sin^{-1}3x\frac{dy}{dx} + y^2\frac{3}{\sqrt{1-9x^2}} = \frac{-9xy}{\sqrt{1-9x^2}}$$

$$\sqrt{1 - 9x^2}\frac{dy}{dx} + \frac{3y^2}{\sqrt{1-9x^2}} + \frac{9xy}{\sqrt{1-9x^2}} = 0$$

$$(1 - 9x^2)\frac{dy}{dx} + 3y^2 + 9xy = 0$$

13. Given that $y = x^n[A\cos(\ln x) + B\sin(\ln x)]$, where A and B are constants, show that

$$x^2\frac{d^2y}{dx^2} + (1 - 2n)x\frac{dy}{dx} + (1 + n^2)y = 0$$

$$y = x^n[A\cos(\ln x) + B\sin(\ln x)] \quad \ldots (1)$$

$$\frac{dy}{dx} = nx^{n-1}[A\cos(\ln x) + B\sin(\ln x)] + x^n\left\{-\frac{A}{x}\sin(\ln x) + \frac{B}{x}\cos(\ln x)\right\}$$

Multiply by x, we have,

$$x\frac{dy}{dx} = nx^n[A\cos(\ln x) + B\sin(\ln x)] + x^n\{-A\sin(\ln x) + B\cos(\ln x)\}$$

By (1),

$$x\frac{dy}{dx} = ny + x^n\{-A\sin(\ln x) + B\cos(\ln x)\} \quad \ldots (2)$$

Differentiate again,

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=n\frac{dy}{dx}+nx^{n-1}\{-A\sin(\ln x)+B\cos(\ln x)\}-x^n\left\{\frac{A}{x}\cos(\ln x)+\frac{B}{x}\sin(\ln x)\right\}$$

Multipy by x,

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=nx\frac{dy}{dx}+nx^n\{-A\sin(\ln x)+B\cos(\ln x)\}-x^n\{A\cos(\ln x)+B\sin(\ln x)\}$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=nx\frac{dy}{dx}+n\left[x\frac{dy}{dx}-ny\right]-y \quad \text{, by (1) and (2).}$$

$$x^2\frac{d^2y}{dx^2}+(1-2n)x\frac{dy}{dx}+(1+n^2)y=0$$

14. Given that $y=\sin^{-1}x$, show that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0$

$$y=\sin^{-1}x \Longrightarrow \sin y = x \Longrightarrow \cos y\frac{dy}{dx}=1 \quad \dots(1)$$

Differentiate (1), $\cos y\frac{d^2y}{dx^2}-\left(\sin y\frac{dy}{dx}\right)\frac{dy}{dx}=0$

Multiply by $\cos y$, $\cos^2 y\frac{d^2y}{dx^2}-\left(\sin y\frac{dy}{dx}\right)\left(\cos y\frac{dy}{dx}\right)=0$

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}(1)=0 \quad \text{, by (1)}$$

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0$$

15. Let $y=5e^{(\sqrt{3}-2)x}+3e^{-(\sqrt{3}+2)x}$, show that $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+y=0.$

Method 1

Let $y=u+v$, where $u=5e^{(\sqrt{3}-2)x}, v=3e^{-(\sqrt{3}+2)x}$.

$$\frac{du}{dx}=5(\sqrt{3}-2)e^{(\sqrt{3}-2)x},\quad \frac{d^2u}{dx^2}=5(\sqrt{3}-2)^2e^{(\sqrt{3}-2)x}=5(7-4\sqrt{3})e^{(\sqrt{3}-2)x}$$

Hence, $\frac{d^2u}{dx^2}+4\frac{du}{dx}+u=5(7-4\sqrt{3})e^{(\sqrt{3}-2)x}+20(\sqrt{3}-2)e^{(\sqrt{3}-2)x}+5e^{(\sqrt{3}-2)x}$

$$=5e^{(\sqrt{3}-2)x}\left[(7-4\sqrt{3})+4(\sqrt{3}-2)+1\right]=0$$

$$\frac{dv}{dx}=-5(\sqrt{3}+2)e^{-(\sqrt{3}+2)x},\quad \frac{d^2v}{dx^2}=5(\sqrt{3}+2)^2e^{-(\sqrt{3}+2)x}=5(7+4\sqrt{3})e^{-(\sqrt{3}+2)x}$$

Hence, $\frac{d^2v}{dx^2}+4\frac{dv}{dx}+v=5(7+4\sqrt{3})e^{-(\sqrt{3}+2)x}-20(\sqrt{3}-2)e^{-(\sqrt{3}+2)x}+5e^{-(\sqrt{3}+2)x}$

$$=5e^{-(\sqrt{3}+2)x}\left[(7+4\sqrt{3})-4(\sqrt{3}+2)+1\right]=0$$

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+y=\frac{d^2(u+v)}{dx^2}+4\frac{d(u+v)}{dx}+(u+v)=\left(\frac{d^2u}{dx^2}+4\frac{du}{dx}+u\right)+\left(\frac{d^2v}{dx^2}+4\frac{dv}{dx}+v\right)=0$$

Method 2

$y = 5e^{(\sqrt{3}-2)x} + 3e^{-(\sqrt{3}+2)x}$

$e^{2x}y = 5e^{\sqrt{3}x} + 3e^{-\sqrt{3}x}$ ... (1)

$e^{2x}\frac{dy}{dx} + 2e^{2x}y = 5\sqrt{3}e^{\sqrt{3}x} - 3\sqrt{3}e^{-\sqrt{3}x}$

$\left(e^{2x}\frac{d^2y}{dx^2} + 2e^{2x}\frac{dy}{dx}\right) + \left(2e^{2x}\frac{dy}{dx} + 4e^{2x}y\right) = 15e^{\sqrt{3}x} + 9\sqrt{3}e^{-\sqrt{3}x}$

$e^{2x}\frac{d^2y}{dx^2} + 4e^{2x}\frac{dy}{dx} + 4e^{2x}y = 3e^{2x}y$, by (1)

$e^{2x}\frac{d^2y}{dx^2} + 4e^{2x}\frac{dy}{dx} + e^{2x}y = 0$

Method 3

Let $\alpha = \sqrt{3} - 2$, $\beta = -(\sqrt{3}+2)$. $\alpha + \beta = -4$, $\alpha\beta = (-2)^2 - 3 = 1$

α, β are roots of $u^2 + 4u + 1 = 0$ ... (1)

Hence $\begin{cases}\alpha^2 + 4\alpha + 1 = 0 \\ \beta^2 + 4\beta + 1 = 0\end{cases}$ ... (2)

$y = 5e^{(\sqrt{3}-2)x} + 3e^{-(\sqrt{3}+2)x} = 5e^{\alpha x} + 3e^{\beta x}$

$\frac{dy}{dx} = 5\alpha e^{\alpha x} + 3\beta e^{\beta x}$, $\frac{d^2y}{dx^2} = 5\alpha^2 e^{\alpha x} + 3\beta^2 e^{\beta x}$

$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 5e^{\alpha x}(\alpha^2 + 4\alpha + 1) + 3e^{\beta x}(\beta^2 + 4\beta + 1) = 0$, by (2).

<u>Comment</u>

(a) The **<u>general</u>** solution of the differential equation: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$ is

$y = c_1 e^{(\sqrt{3}-2)x} + c_2 e^{-(\sqrt{3}+2)x}$, c_1, c_2 are integrating constants.

So $y = 5e^{(\sqrt{3}-2)x} + 3e^{-(\sqrt{3}+2)x}$ is a **<u>special</u>** solution of the differential equation.

(b) It is interesting that solving $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$ is just to find the roots of the auxiliary equation $u^2 + 4u + 1 = 0$, that is $\alpha = \sqrt{3} - 2$, $\beta = -(\sqrt{3}+2)$ and then form the **general** solution $y = c_1 e^{(\sqrt{3}-2)x} + c_2 e^{-(\sqrt{3}+2)x}$. Solving this kind of second order differential equation becomes solving a quadratic equation. The story is longer since quadratic equation may have equal roots or complex roots and is not discussed here.

(c) **<u>Showing</u>** differential equation is just learning how to dive in the sea of calculus and **<u>solving</u>** differential equation is to get some pearls in the sea.

19/1/2018

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